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A note on compact operators on normed linear spaces

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Abstract

We show that there is no surjective compact operator on a normed linear infinite-dimensional space.

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Throughout the paper, by a space we mean a normed linear space over real or complex numbers and B_X is reserved for the open unit ball of a space X. Then the open ball of radius $r > 0$ and center x in a space X is $x + rB_X$.

The starting point of our remark is the Open Mapping Theorem (see [1, Theorem 2.11]). This fundamental principle of functional analysis states that any bounded linear operator from a Banach space onto another Banach space is open, i.e., it maps open sets to open sets. In particular, it implies that there is no compact operator from a Banach space onto a Banach space provided the target space is infinite-dimensional. (We recall that a bounded linear operator $T : X \to Y$ from a space X to a space Y is compact if $\overline{T(M)}$ is compact in Y for any bounded set $M \subset X$.) The reason is that infinite-dimensional spaces have no nonempty open sets with compact closure (see [1, Theorem 1.22]). Nevertheless, if we omit the requirement of completeness, the assertion is no longer true.

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Example. There exist infinite-dimensional spaces X, Y and a compact surjective operator $T : X \to Y$.

Proof. Let $K = \{(x_n) \in \ell_2 : |x_n| \leqslant 1/n\}$ (we recall that ℓ^2 is the space of all real or complex sequences (x_n) satisfying $\sum_n |x_n|^2 < \infty$ with the norm $\|(x_n)\| = (\sum_n |x_n|^2)^{1/2}$) and let $X = \bigcup_n nK$ with the norm defined as the Minkowski functional of K (see [1, Definitions 1.33]). Let $Y = X$ with the original ℓ^2 norm and $T : X \to Y$ be the identity mapping. Then both the spaces X and Y are infinite-dimensional and T is a compact surjective operator. □

From the point of view of the previous example it might be of some interest that such an operator cannot be constructed in case $X = Y$.

Theorem. *Let X be an infinite-dimensional space and $T : X \to X$ be a compact operator. Then T is not surjective.*

Proof. Assume that $T : X \to X$ is a compact surjective operator on an infinite-dimensional space X. Then the set $K = \overline{T(B_X)}$ is compact and $X = \bigcup_n nK$. Since

$$K \subset X = T(X) = \bigcup_{n=1}^{\infty} T(nK)$$

and the sets $T(nK)$, $n \in \mathbb{N}$, are compact, by the Baire Category Theorem (see [1, Theorem 2.2]) there exists $n \in \mathbb{N}$ such that the set $K \cap T(nK)$ has nonempty interior in K, i.e., there exists an open set $U \subset X$ such that

$$\emptyset \neq K \cap U \subset K \cap T(nK). \tag{1}$$

Since K is the closure of $T(B_X)$, we can find $y \in T(B_X)$ and $r > 0$ such that $y + rB_X \subset U$. We pick $x \in B_X$ with $Tx = y$ and use continuity of T to choose $s > 0$ such that

$$x + sB_X \subset B_X \quad \text{and} \quad T(sB_X) \subset rB_X. \tag{2}$$

By (1) and (2)

$$T(x + sB_X) \subset T(B_X) \cap (y + rB_X) \subset T(nK). \tag{3}$$

If $\operatorname{Ker} T$ denotes the space $T^{-1}(0)$, we consider the quotient space $\widehat{X} = X/\operatorname{Ker} T$ (see [1, Definitions 1.40]) and the operator $\widehat{T} : \widehat{X} \to X$ defined as

$$\widehat{T}(qx) = Tx, \quad x \in X,$$

where $q : X \to \widehat{X}$ is the quotient mapping. Then $\widehat{X}$ is infinite-dimensional space (otherwise codimension of $\operatorname{Ker} T$ would be finite and X itself would be finite-dimensional as T is onto) and $\widehat{T}$ is a compact operator.

Indeed, the quotient mapping q satisfies $q(B_X) = B_{\widehat{X}}$ (see the proof of [1, Theorem 1.41]) and thus the closure of the set

$$\widehat{T}(B_{\widehat{X}}) = \widehat{T}(qB_X) = T(B_X)$$

is compact in X. Finally, inclusion (3) yields

$$\begin{aligned}\widehat{T}(qx + sB_{\widehat{X}}) &= \widehat{T}(q(x + sB_X)) \\ &= T(x + sB_X) \\ &\subset T(nK) \\ &= \widehat{T}(q(nK)). \end{aligned} \tag{4}$$

Since $\widehat{T}$ is injective, (4) implies

$$qx + sB_{\widehat{X}} \subset q(nK).$$

As the set $q(nK)$ is compact in $\widehat{X}$ (q is continuous), we have found a nonempty ball in an infinite-dimensional space contained in a compact set, an obvious contradiction (see [1, Theorem 1.22]). This finishes the proof. □

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Reference

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